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On certain two-dimensional Schlömilch series

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Abstract. By using representations for one-dimensional (1D) Schlömilch series, Lommel's expansion for Bessel functions of the first kind, and elementary series manipulations, several algebraic formulae for two-dimensional (2D) Schlömilch series are derived. One of these formulae contains as a special case an identity involving a certain trigonometric lattice sum conjectured by Henkel and Weston which has recently been derived by Boersma and de Doelder using the 2D Poisson summation formula. The latter method is used to give an alternative derivation for 2D Schlömilch series. Further, it is shown that representation by 2D Schlömilch series of a null-function is not unique. In addition, representations for several other 1D type series are given.

1. Introduction

Series of the type

$$\sum_{m=1}^{\infty} a_m \frac{J_\nu(mx)}{m^\nu}$$

are called one-dimensional (1D) Schlömilch series; and a two-dimensional (2D) Schlömilch series may be defined by

$$\mathscr{S}_\nu(x) \equiv \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu}. \tag{1.1}$$

If the coefficients a_{mn} are functions whose arguments are linear or quadratic in the summation indices, this sum is of a type known as a lattice sum. Such sums are important, for example, in the calculation of physical properties of crystal surfaces [1]. We note that if the absolute value of the (complex) coefficients appearing respectively in the 1D and 2D Schlömilch series are bounded by a constant, then for $x>0$ the 1D series converges for $\mathrm{Re}\,\nu > -\frac{1}{2}$ and the 2D series converges for $\mathrm{Re}\,\nu>0$. In addition, the 2D series converges absolutely for $\mathrm{Re}\,\nu>\frac{3}{2}$ and the 1D series converges absolutely for $\mathrm{Re}\,\nu>\frac{1}{2}$.

Thus, the sums defined below are 2D Schlömilch lattice sums:

$$R_\nu(x) \equiv \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu} \tag{1.2}$$

$$S_\nu(x) \equiv \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu} \tag{1.3}$$

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$$T_\nu(x) \equiv \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^m \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu} \tag{1.4}$$

$$U_\nu(x) \equiv \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu} \tag{1.5}$$

where $x>0$ and the order ν of the Bessel function is such that Re $\nu>0$. In this paper we shall derive closed form expressions for the sums defined by equations (1.2)–(1.5) and somewhat more general sums in terms of elementary algebraic functions.

As a byproduct of this investigation we shall be able to show in addition (in section 5) that the null-function on the open interval $(0, \pi/2)$ may be represented in two different ways by series of the type defined by equation (1.1). In section 4, representations for several other 1D type series are given.

2. Evaluation of $R_\nu(x)$

We write equation (1.2) as

$$R_\nu(x) = \sum_{m=1}^{\infty} (-1)^m \frac{J_\nu(2mx)}{m^\nu} + \hat{R}_\nu(x) \tag{2.1}$$

where

$$\hat{R}_\nu(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu}. \tag{2.2}$$

It is well known from Lommel's expansion for Bessel functions [2, p 140] that

$$\frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu} = \sum_{r=0}^{\infty} \frac{(-xm^2)^r}{r!} \frac{J_{\nu+r}(2nx)}{n^{\nu+r}} \tag{2.3}$$

which is just a special case of the addition theorem [3, p 24] for generalized Gaussian hypergeometric functions. Thus, using equation (2.3) together with equation (2.2) yields

$$\hat{R}_\nu(x) = \sum_{m=1}^{\infty} (-1)^m \sum_{r=0}^{\infty} \frac{(-xm^2)^r}{r!} \sum_{n=1}^{\infty} (-1)^n \frac{J_{\nu+r}(2nx)}{n^{\nu+r}} \tag{2.4}$$

where we have interchanged the second and third summations.

We shall need Nielsen's summation formula [2, p 636, equation (4)], [4, p 678, equation (11)] for $x>0$:

$$\sum_{m=1}^{\infty} (-1)^m \frac{J_\nu(2mx)}{m^\nu} = -\frac{x^\nu}{2\Gamma(1+\nu)} + \frac{\sqrt{\pi}x^{-\nu}}{\Gamma(\frac{1}{2}+\nu)} \sum_{m=1}^{p} [x^2-(m-\tfrac{1}{2})^2\pi^2]^{\nu-(1/2)} \tag{2.5}$$

where Re $\nu > -\frac{1}{2}$ and p is a non-negative integer such that

$$p-\tfrac{1}{2} < x/\pi < p+\tfrac{1}{2}. \tag{2.6}$$

The series on the left side of equation (2.5) is a 1D Schlömilch series; here when $0<x<\pi/2$, then $p=0$ and the second term vanishes by definition. Thus, using equation (2.5) together with equation (2.4) we see that for Re $\nu>-\frac{1}{2}$, $x>0$

$$\hat{R}_\nu(x)=\sum_{m=1}^{\infty}(-1)^m\sum_{r=0}^{\infty}\frac{(-xm^2)^r}{r!}$$

$$\times\left\{-\frac{x^{\nu+r}}{2\Gamma(1+\nu+r)}+\frac{\sqrt{\pi}x^{-\nu-r}}{\Gamma(\frac{1}{2}+\nu+r)}\sum_{s=1}^{p}[x^2-(s-\tfrac{1}{2})^2\pi^2]^{\nu+r-(1/2)}\right\}$$

$$=-\frac{x^\nu}{2\Gamma(1+\nu)}\sum_{m=1}^{\infty}(-1)^m\sum_{r=0}^{\infty}\frac{(-x^2m^2)^r}{(1+\nu)_r r!}+\frac{\sqrt{\pi}x^{-\nu}}{\Gamma(\frac{1}{2}+\nu)}\sum_{s=1}^{p}[x^2-(s-\tfrac{1}{2})^2\pi^2]^{\nu-(1/2)}$$

$$\times\sum_{m=1}^{\infty}(-1)^m\sum_{r=0}^{\infty}\frac{(-m^2[x^2-(s-\frac{1}{2})^2\pi^2])^r}{(\frac{1}{2}+\nu)_r r!}$$

where in the second term we have interchanged summations.

Since it is immediately clear that the r-summations above can be expressed as Bessel functions $J_\nu(\cdot)$ and $J_{\nu-(1/2)}(\cdot)$, the latter result may be rewritten as

$$\hat{R}_\nu(x)=-\frac{1}{2}\sum_{m=1}^{\infty}(-1)^m\frac{J_\nu(2mx)}{m^\nu}+\Gamma(\tfrac{1}{2})x^{-\nu}\sum_{s=1}^{p}(\sqrt{x^2-(s-\tfrac{1}{2})^2\pi^2})^{\nu-(1/2)}$$

$$\times\sum_{m=1}^{\infty}(-1)^m\frac{J_{\nu-(1/2)}(2m\sqrt{x^2-(s-\frac{1}{2})^2\pi^2})}{m^{\nu-(1/2)}} \qquad (2.7)$$

where Re $\nu>-\frac{1}{2}$ and p is a non-negative integer such that

$$p-\tfrac{1}{2}<x/\pi<p+\tfrac{1}{2}.$$

Now by applying the result for summation of the 1D Schlömilch series in equation (2.5) a second time to the second summation over m in equation (2.7) we have

$$\sum_{m=1}^{\infty}(-1)^m\frac{J_{\nu-(1/2)}(2m\sqrt{x^2-(s-\frac{1}{2})^2\pi^2})}{m^{\nu-(1/2)}}$$

$$=-\frac{(\sqrt{x^2-(s-\frac{1}{2})^2\pi^2})^{\nu-(1/2)}}{2\Gamma(\frac{1}{2}+\nu)}+\frac{\Gamma(\frac{1}{2})}{\Gamma(\nu)}(\sqrt{x^2-(s-\tfrac{1}{2})^2\pi^2})^{(1/2)-\nu}$$

$$\times\sum_{t=1}^{q}[x^2-(s-\tfrac{1}{2})^2\pi^2-(t-\tfrac{1}{2})^2\pi^2]^{\nu-1} \qquad (2.8)$$

where Re $\nu>0$ and q is an integer such that

$$q-\tfrac{1}{2}<\sqrt{x^2/\pi^2-(s-\tfrac{1}{2})^2}<q+\tfrac{1}{2}. \qquad (2.9)$$

Thus, combining equations (2.1), (2.7) and (2.8) yields

$$R_\nu(x)=\frac{1}{2}\sum_{m=1}^{\infty}(-1)^m\frac{J_\nu(2mx)}{m^\nu}-\frac{\Gamma(\frac{1}{2})x^{-\nu}}{2\Gamma(\frac{1}{2}+\nu)}\sum_{s=1}^{p}[x^2-(s-\tfrac{1}{2})^2\pi^2]^{\nu-(1/2)}$$

$$+\frac{\pi x^{-\nu}}{\Gamma(\nu)}\sum_{s=1}^{p}\sum_{t=1}^{q}[x^2-(s-\tfrac{1}{2})^2\pi^2-(t-\tfrac{1}{2})^2\pi^2]^{\nu-1} \qquad (2.10)$$

so that on noting once again equation (2.5) we deduce

$$R_\nu(x)=-\frac{x^\nu}{4\Gamma(1+\nu)}+\frac{\pi x^{-\nu}}{\Gamma(\nu)}\sum_{s=1}^{p}\sum_{t=1}^{q}[x^2-(s-\tfrac{1}{2})^2\pi^2-(t-\tfrac{1}{2})^2\pi^2]^{\nu-1} \qquad (2.11)$$

where Re $\nu>0$, $x>0$.

We note that the expression $x^2-(s-\frac{1}{2})^2\pi^2$ in equations (2.7), (2.8) and (2.10) no longer appears by itself in the above result so that the values of p and q are determined by means of the inequality (2.9). Thus, we have that p and $q(s)$ are the largest integers such that

$$p<\frac{1}{2}+\sqrt{\frac{x^2}{\pi^2}-\frac{1}{4}} \qquad q(s)<\frac{1}{2}+\sqrt{\frac{x^2}{\pi^2}-(s-\tfrac{1}{2})^2}$$

where $s=1,2,\ldots,p$. Equation (2.11) may be rewritten more compactly as

$$R_\nu(x)=-\frac{x^\nu}{4\Gamma(1+\nu)}+\frac{\pi^{2\nu-1}}{2^{2\nu-2}}\frac{x^{-\nu}}{\Gamma(\nu)}\sum_{s,t\text{ odd}}^{s^2+t^2<4x^2/\pi^2}\left(\frac{4x^2}{\pi^2}-s^2-t^2\right)^{\nu-1} \tag{2.12}$$

where Re $\nu>0$, $x>0$. This evidently completes the derivation of the algebraic representation for $R_\nu(x)$.

We remark that when $0<x<\pi/\sqrt{2}$, there is no contribution from the double sum in equations (2.11) and (2.12). This is easy to see from equation (2.12), since in this case there are no odd integers s and t such that $s^2+t^2<2$ (cf equation (5.2)).

3. A trigonometric lattice sum

In [5, p 497] Boersma and de Doelder deduced a closed form elementary formula for the trigonometric lattice sum

$$S(x)\equiv\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\frac{\sin(2x\sqrt{m^2+n^2})}{\sqrt{m^2+n^2}}$$

which occurs in finite-size scaling of the three-dimensional spherical model of ferromagnetism [6]. Since

$$\frac{\sin z}{z}=\sqrt{\frac{\pi}{2}}\frac{J_{1/2}(z)}{\sqrt{z}}$$

we see that

$$S(x)=\sqrt{\pi x}\,R_{1/2}(x).$$

Thus, setting $\nu=\frac{1}{2}$ in equation (2.12) gives for $x>0$

$$S(x)=-\frac{x}{2}+2\sum_{s,t\text{ odd}}^{s^2+t^2<4x^2/\pi^2}\left(\frac{4x^2}{\pi^2}-s^2-t^2\right)^{-(1/2)}$$

which (except for a misprint) is derived in [5] by essentially employing the two-dimensional Poisson summation formula. For $|x|<\pi/\sqrt{2}$, we see that $S(x)=-x/2$ which was conjectured by Henkel and Weston.

4. Evaluation of $S_\nu(x)$, $T_\nu(x)$, $U_\nu(x)$

Define for $x>0$

$$\hat{S}_\nu(x) \equiv \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu} \tag{4.1}$$

so that

$$S_\nu(x) = \sum_{m=1}^{\infty} \frac{J_\nu(2mx)}{m^\nu} + \hat{S}_\nu(x). \tag{4.2}$$

The above 1D Schlömilch series for positive x is given by [4, p 678, equation (11)]

$$\sum_{m=1}^{\infty} \frac{J_\nu(2mx)}{m^\nu} = -\frac{x^\nu}{2\Gamma(1+\nu)} + \frac{\sqrt{\pi}x^{\nu-1}}{2\Gamma(\frac{1}{2}+\nu)} + \frac{\sqrt{\pi}x^{-\nu}}{\Gamma(\frac{1}{2}+\nu)} \sum_{m=1}^{p'} (x^2-\pi^2m^2)^{\nu-(1/2)} \tag{4.3}$$

where Re $\nu > -\frac{1}{2}$ and p' is a non-negative integer such that

$$p' < x/\pi < p'+1. \tag{4.4}$$

Since the evaluation of $S_\nu(x)$ is similar to that of $R_\nu(x)$, we only sketch the details. Thus, by using equation (2.3) we have from equation (4.1)

$$\hat{S}_\nu(x) = \sum_{m=1}^{\infty} \sum_{r=0}^{\infty} \frac{(-xm^2)^r}{r!} \sum_{n=1}^{\infty} \frac{J_{\nu+r}(2nx)}{n^{\nu+r}}.$$

Now substituting the expression for the 1D Schlömilch series in equation (4.3) with ν replaced by $\nu+r$ into this result yields after simplification

$$\hat{S}_\nu(x) = -\frac{1}{2} \sum_{m=1}^{\infty} \frac{J_\nu(2mx)}{m^\nu} + \frac{1}{2}\sqrt{\frac{\pi}{x}} \sum_{m=1}^{\infty} \frac{J_{\nu-(1/2)}(2mx)}{m^{\nu-(1/2)}}$$

$$+\frac{\sqrt{\pi}}{x^\nu} \sum_{s=1}^{p'} (\sqrt{x^2-\pi^2s^2})^{\nu-(1/2)} \sum_{m=1}^{\infty} \frac{J_{\nu-(1/2)}(2m\sqrt{x^2-\pi^2s^2})}{m^{\nu-(1/2)}} \tag{4.5}$$

where Re $\nu > -\frac{1}{2}$ and the inequality (4.4) holds.

Next, combining equations (4.2) and (4.5) and applying equation (4.3) to the three resulting summations with index m yields after simplification

$$S_\nu(x) = -\frac{x^\nu}{4\Gamma(1+\nu)} + \frac{\pi}{4}\frac{x^{\nu-2}}{\Gamma(\nu)} + \frac{\pi^{2\nu-1}x^{-\nu}}{\Gamma(\nu)} \sum_{s=0}^{p'} \sum_{t=1}^{q'(s)} \left(\frac{x^2}{\pi^2}-s^2-t^2\right)^{\nu-1} \tag{4.6}$$

where Re $\nu>0, x>0$ and $p', q'(s)$ are the largest integers such that $p'<x/\pi$, $q'(s)<\sqrt{x^2/\pi^2-s^2}$ ($s=0, 1, \ldots, p'$), $q'(0)=p'$. It is easy to see that we may also write equation (4.6)

$$S_\nu(x) = -\frac{x^\nu}{4\Gamma(1+\nu)} + \frac{\pi}{4}\frac{x^{\nu-2}}{\Gamma(\nu)} + \frac{\pi^{2\nu-1}x^{-\nu}}{\Gamma(\nu)} \sum_{s=0,t=1}^{s^2+t^2<x^2/\pi^2} \left(\frac{x^2}{\pi^2}-s^2-t^2\right)^{\nu-1} \tag{4.7}$$

where Re $\nu>0$, $x>0$.

Formulae for $T_\nu(x)$ and $U_\nu(x)$ are derived similarly; and once either $T_\nu(x)$ or $U_\nu(x)$ is known, then so is the other since it is easy to verify for positive x that

$$U_\nu(x)-T_\nu(x)=\frac{\sqrt{\pi}x^{\nu-1}}{2\Gamma(\frac{1}{2}+\nu)}+\frac{\pi^{2\nu-(1/2)}x^{-\nu}}{\Gamma(\frac{1}{2}+\nu)}\times\left\{\sum_{m=1}^{p'}\left[\frac{x^2}{\pi^2}-m^2\right]^{\nu-(1/2)}-\sum_{m=1}^{p}\left[\frac{x^2}{\pi^2}-(m-\tfrac{1}{2})^2\right]^{\nu-(1/2)}\right\} \tag{4.8}$$

where Re $\nu>-\frac{1}{2}$ and p, p' are non-negative integers such that

$$\begin{aligned} &p-\tfrac{1}{2}<x/\pi<p+\tfrac{1}{2}\\ &p'<x/\pi<p'+1. \end{aligned} \tag{4.9}$$

Thus,

$$\begin{aligned} U_\nu(x)=&\frac{\sqrt{\pi}x^{\nu-1}}{4\Gamma(\frac{1}{2}+\nu)}-\frac{x^\nu}{4\Gamma(1+\nu)}-\frac{\pi^{2\nu-(1/2)}x^{-\nu}}{2\Gamma(\frac{1}{2}+\nu)}\\ &\times\left\{\sum_{m=1}^{p}\left[\frac{x^2}{\pi^2}-(m-\tfrac{1}{2})^2\right]^{\nu-(1/2)}-\sum_{m=1}^{p'}\left[\frac{x^2}{\pi^2}-m^2\right]^{\nu-(1/2)}\right\}\\ &+\frac{\pi^{2\nu-1}x^{-\nu}}{\Gamma(\nu)}\left\{\frac{1}{2}\sum_{m=1}^{p}\left[\frac{x^2}{\pi^2}-(m-\tfrac{1}{2})^2\right]^{\nu-1}+\sum_{s,t=1}^{\xi^2<x^2/\pi^2}\left[\frac{x^2}{\pi^2}-\xi^2\right]^{\nu-1}\right\} \end{aligned} \tag{4.10}$$

and by using this together with equation (4.8) gives

$$\begin{aligned} T_\nu(x)=&-\frac{\sqrt{\pi}x^{\nu-1}}{4\Gamma(\frac{1}{2}+\nu)}-\frac{x^\nu}{4\Gamma(1+\nu)}+\frac{\pi^{2\nu-(1/2)}x^{-\nu}}{2\Gamma(\frac{1}{2}+\nu)}\\ &\times\left\{\sum_{m=1}^{p}\left[\frac{x^2}{\pi^2}-(m-\tfrac{1}{2})^2\right]^{\nu-(1/2)}-\sum_{m=1}^{p'}\left[\frac{x^2}{\pi^2}-m^2\right]^{\nu-(1/2)}\right\}\\ &+\frac{\pi^{2\nu-1}x^{-\nu}}{\Gamma(\nu)}\left\{\frac{1}{2}\sum_{m=1}^{p}\left[\frac{x^2}{\pi^2}-(m-\tfrac{1}{2})^2\right]^{\nu-1}+\sum_{s,t=1}^{\xi^2<x^2/\pi^2}\left[\frac{x^2}{\pi^2}-\xi^2\right]^{\nu-1}\right\} \end{aligned} \tag{4.11}$$

where Re $\nu>0$, $x>0$, p, p' are determined by the inequalities (4.9) and ξ^2 is defined by $\xi^2=s^2+(t-\frac{1}{2})^2$.

We observe that equations (2.12), (4.7), (4.10), (4.11) may be expressed in a unified and more transparent form; namely for $x>0$, Re $\nu>0$:

$$R_\nu(x)=-\frac{x^\nu}{4\Gamma(1+\nu)}+\frac{\pi^{2\nu-1}x^{-\nu}}{2^{2\nu}\Gamma(\nu)}\sum_{\substack{s,t\in\mathbb{Z}\\ s,t\text{ odd}}}^{s^2+t^2<4x^2/\pi^2}(4x^2/\pi^2-s^2-t^2)^{\nu-1}$$

$$S_\nu(x)=-\frac{x^\nu}{4\Gamma(1+\nu)}+\frac{\pi^{2\nu-1}x^{-\nu}}{2^{2\nu}\Gamma(\nu)}\sum_{\substack{s,t\in\mathbb{Z}\\ s,t\text{ even}}}^{s^2+t^2<4x^2/\pi^2}(4x^2/\pi^2-s^2-t^2)^{\nu-1}$$

$$\begin{aligned} T_\nu(x)=&-\frac{x^\nu}{4\Gamma(1+\nu)}-\frac{\pi^{2\nu-(1/2)}x^{-\nu}}{2^{2\nu+1}\Gamma(\frac{1}{2}+\nu)}\sum_{m\in\mathbb{Z}}^{m^2<4x^2/\pi^2}(-1)^m(4x^2/\pi^2-m^2)^{\nu-(1/2)}\\ &+\frac{\pi^{2\nu-1}x^{-\nu}}{2^{2\nu}\Gamma(\nu)}\sum_{\substack{s,t\in\mathbb{Z}\\ s\text{ even},t\text{ odd}}}^{s^2+t^2<4x^2/\pi^2}(4x^2/\pi^2-s^2-t^2)^{\nu-1} \end{aligned}$$

$$U_\nu(x) = -\frac{x^\nu}{4\Gamma(1+\nu)} + \frac{\pi^{2\nu-(1/2)}x^{-\nu}}{2^{2\nu+1}\Gamma(\frac{1}{2}+\nu)} \sum_{m\in\mathbb{Z}}^{m^2<4x^2/\pi^2} (-1)^m(4x^2/\pi^2 - m^2)^{\nu-(1/2)}$$
$$+\frac{\pi^{2\nu-1}x^{-\nu}}{2^{2\nu}\Gamma(\nu)} \sum_{\substack{s,t\in\mathbb{Z}\\ s\text{ even}, t\text{ odd}}}^{s^2+t^2<4x^2/\pi^2} (4x^2/\pi^2 - s^2 - t^2)^{\nu-1}$$

where $\mathbb{Z}$ is the set of all integers (positive, negative, zero).

Finally, we note that by using equations (2.5) and (4.3) computations similar to those that gave $R_\nu(x)$ and $S_\nu(x)$ yield for $x>0$, Re $\nu > -\frac{1}{2}$ and arbitrary complex β representations for the following 1D type series:

$$\sum_{m=1}^{\infty} (-1)^m \frac{J_\nu(2x\sqrt{\beta^2+m^2})}{(\sqrt{\beta^2+m^2})^\nu}$$
$$= -\frac{J_\nu(2\beta x)}{2\beta^\nu} + \frac{\sqrt{\pi}x^{-\nu}}{\beta^{\nu-(1/2)}}$$
$$\times \sum_{m=1}^{p} (\sqrt{x^2-(m-\tfrac{1}{2})^2\pi^2})^{\nu-(1/2)} J_{\nu-(1/2)}(2\beta\sqrt{x^2-(m-\tfrac{1}{2})^2\pi^2})$$

and

$$\sum_{m=1}^{\infty} \frac{J_\nu(2x\sqrt{\beta^2+m^2})}{(\sqrt{\beta^2+m^2})^\nu}$$
$$= -\frac{J_\nu(2\beta x)}{2\beta^\nu} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\frac{J_{\nu-(1/2)}(2\beta x)}{\beta^{\nu-(1/2)}}$$
$$+\frac{\sqrt{\pi}x^{-\nu}}{\beta^{\nu-(1/2)}} \sum_{m=1}^{p'} (\sqrt{x^2-m^2\pi^2})^{\nu-(1/2)} J_{\nu-(1/2)}(2\beta\sqrt{x^2-m^2\pi^2})$$

where $p-\frac{1}{2} < x/\pi < p+\frac{1}{2}$, $p' < x/\pi < p'+1$. These yield the known results for Re $\nu > -\frac{1}{2}$

$$\sum_{m=1}^{\infty} (-1)^m \frac{J_\nu(2x\sqrt{\beta^2+m^2})}{(\sqrt{\beta^2+m^2})^\nu} = -\frac{J_\nu(2\beta x)}{2\beta^\nu} \qquad 0<x<\pi/2$$

$$\sum_{m=1}^{\infty} \frac{J_\nu(2x\sqrt{\beta^2+m^2})}{(\sqrt{\beta^2+m^2})^\nu} = -\frac{J_\nu(2\beta x)}{2\beta^\nu} + \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\frac{J_{\nu-(1/2)}(2\beta x)}{\beta^{\nu-(1/2)}} \qquad 0<x<\pi$$

which are found in [4, section 5.7.22(3), p 682] where apparently there appears the misprint Re $\nu \geqslant 0$.

5. Null-functions as 2D Schlömilch series

In [2, section 19.41] Watson discusses null-functions expressed as 1D Schlömilch series. For example, from equation (2.5) for Re $\nu > -\frac{1}{2}$, $0<x<\pi/2$

$$\sum_{m=1}^{\infty} (-1)^m \frac{J_\nu(2mx)}{(mx)^\nu} + \frac{1}{2\Gamma(1+\nu)} = 0 \tag{5.1}$$

where the constant term outside the summation is regarded as the first term of the series.

The 2D series defined by equations (1.2)–(1.5) may be regarded as generalizations of the 1D Schlömilch series given by equations (2.5) and (4.3). From equations (2.12), (4.7), (4.10) and (4.11) respectively we obtain for Re $\nu>0$ the following:

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{m+n}\frac{J_\nu(2x\sqrt{m^2+n^2})}{(x\sqrt{m^2+n^2})^\nu}+\frac{1}{4\Gamma(1+\nu)}=0 \qquad 0<x<\pi/\sqrt{2} \tag{5.2}$$

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{J_\nu(2x\sqrt{m^2+n^2})}{(x\sqrt{m^2+n^2})^\nu}+\frac{1}{4\Gamma(1+\nu)}-\frac{\pi x^{-2}}{4\Gamma(\nu)}=0 \qquad 0<x<\pi$$

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{m}\frac{J_\nu(2x\sqrt{m^2+n^2})}{(x\sqrt{m^2+n^2})^\nu}+\frac{1}{4\Gamma(1+\nu)}+\frac{\sqrt{\pi}x^{-1}}{4\Gamma(\frac{1}{2}+\nu)}=0 \qquad 0<x<\pi/2 \tag{5.3}$$

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{n}\frac{J_\nu(2x\sqrt{m^2+n^2})}{(x\sqrt{m^2+n^2})^\nu}+\frac{1}{4\Gamma(1+\nu)}-\frac{\sqrt{\pi}x^{-1}}{4\Gamma(\frac{1}{2}+\nu)}=0 \qquad 0<x<\pi/2\,. \tag{5.4}$$

Adding equations (5.3) and (5.4) yields in addition

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m+(-1)^n}{2}\,\frac{J_\nu(2x\sqrt{m^2+n^2})}{(x\sqrt{m^2+n^2})^\nu}+\frac{1}{4\Gamma(1+\nu)}=0 \qquad 0<x<\pi/2\,. \tag{5.5}$$

Guided by the 1D result equation (5.1), we see that only equations (5.2) and (5.5) provide representations for null-functions by 2D Schlömilch series, where also the constant term outside the summation is regarded as the first term of the series. Thus we deduce that representation of a null-function on the interval $0<x<\pi/2$ by a function in the class $\mathscr{S}_\nu(x)$ defined by equation (1.1) is not unique.

Equation (5.2) has been obtained by Allen and Pathria [7] who show in fact that the result may be generalized to certain higher dimensional Schlömilch series. Grosjean [8] has obtained equation (2.11) as well by using Hankel transforms.

6. Alternative evaluation of 2D series

Consider the 2D Schlömilch series for $x>0$, Re $\nu>0$

$$R_\nu(p,q;x)\equiv\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{mp+nq}\frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu}. \tag{6.1}$$

Thus, the series defined previously by $R_\nu(x)$, $S_\nu(x)$, $T_\nu(x)$, $U_\nu(x)$ are just special cases of $R_\nu(p,q;x)$ corresponding respectively to $(p,q)=(1,1)$, $(0,0)$, $(1,0)$, $(0,1)$.

The series given in equation (6.1) is related to the doubly infinite, double series

$$W_\nu(p,q;x)\equiv\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(-1)^{mp+nq}\frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu}. \tag{6.2}$$

Indeed, by properly splitting up the summation domain $\mathbb{Z}\times\mathbb{Z}$, one has

$$W_\nu(p,q;x)=4R_\nu(p,q;x)+\frac{x^\nu}{\Gamma(\nu+1)}-2\sum_{m=1}^{\infty}(-1)^{mp}\frac{J_\nu(2mx)}{m^\nu}+2\sum_{n=1}^{\infty}(-1)^{nq}\frac{J_\nu(2nx)}{n^\nu}$$

so that

$$R_\nu(p,q;x)=-\frac{x^\nu}{4\Gamma(1+\nu)}+\tfrac{1}{4}W_\nu(p,q;x)$$
$$+\tfrac{1}{2}\sum_{m=1}^{\infty}(-1)^{mp}\frac{J_\nu(2mx)}{m^\nu}-\tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{nq}\frac{J_\nu(2nx)}{n^\nu}. \qquad (6.3)$$

The 1D Schlömilch series in equation (6.3) are known from equations (2.5) and (4.3) for the cases $p, q=1$ and $p, q=0$ respectively. Notice also that the 1D series cancel if $p=q$, corresponding to the cases of $R_\nu(x)$ and $S_\nu(x)$.

Alternatively, the 1D Schlömilch series can be evaluated by means of the 1D Poisson summation formula, yielding for $x>0$, $\operatorname{Re}\nu>-\frac{1}{2}$

$$\sum_{m=1}^{\infty}(-1)^{mp}\frac{J_\nu(2mx)}{m^\nu}$$
$$=-\frac{x^\nu}{2\Gamma(\nu+1)}+\frac{\pi^{2\nu-(1/2)}x^{-\nu}}{2^{2\nu}\Gamma(\nu+\frac{1}{2})}$$
$$\times\sum_{k\in\mathbb{Z}}^{(2k+p)^2<4x^2/\pi^2}[4x^2/\pi^2-(2k+p)^2]^{\nu-(1/2)}. \qquad (6.4)$$

To evaluate $W_\nu(p,q;x)$, the 2D Poisson summation formula is employed, as in [5]. Thus, introduce the function

$$f(x,y;t)\equiv\begin{cases}J_\nu(2t\sqrt{x^2+y^2})/(\sqrt{x^2+y^2})^\nu & (x,y)\neq(0,0)\\ t^\nu/\Gamma(\nu+1) & (x,y)=(0,0)\end{cases}$$

and determine its 2D Fourier transform:

$$\mathscr{F}[f(x,y;t)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{J_\nu(2t\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^\nu}\,e^{i\xi x+i\eta y}\,dx\,dy$$
$$=2\pi\int_0^\infty\frac{J_\nu(2t\rho)}{\rho^\nu}J_0(\rho\sqrt{\xi^2+\eta^2})\rho\,d\rho$$
$$=\begin{cases}2\pi\dfrac{2^{1-2\nu}}{\Gamma(\nu)}\dfrac{(4t^2-\xi^2-\eta^2)^{\nu-1}}{t^\nu} & \xi^2+\eta^2<4t^2\\ 0 & \xi^2+\eta^2>4t^2.\end{cases} \qquad (6.5)$$

Here the latter discontinuous integral exists for $\operatorname{Re}\nu>0$ and may be evaluated, for example, by using [9, section 6.574(1), (3)].

Inversion of the Fourier transform equation (6.5) yields

$$f(x,y;t)=\frac{1}{2\pi}\frac{2^{1-2\nu}t^{-\nu}}{\Gamma(\nu)}\iint_{\xi^2+\eta^2<4t^2}(4t^2-\xi^2-\eta^2)^{\nu-1}\,e^{-ix\xi-iy\eta}\,d\xi\,d\eta.$$

Hence, by setting $x=m$, $y=n$ and replacing t by x we obtain

$$\frac{J_\nu(2x\sqrt{m^2+n^2})}{(\sqrt{m^2+n^2})^\nu}=\frac{1}{2\pi}\frac{2^{1-2\nu}x^{-\nu}}{\Gamma(\nu)}\iint\limits_{\xi^2+\eta^2<4x^2}(4x^2-\xi^2-\eta^2)^{\nu-1}\,\mathrm{e}^{-\mathrm{i}m\xi-\mathrm{i}n\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

The latter representation is inserted into the series in equation (6.2) and a formal term-by-term integration is applied, yielding

$$W_\nu(p,q;x)=\frac{1}{2\pi}\frac{2^{1-2\nu}x^{-\nu}}{\Gamma(\nu)}\iint\limits_{\xi^2+\eta^2<4x^2}(4x^2-\xi^2-\eta^2)^{\nu-1}$$
$$\times\sum_{m\in\mathbf{Z}}(-1)^{mp}\,\mathrm{e}^{-\mathrm{i}m\xi}\sum_{n\in\mathbf{Z}}(-1)^{nq}\,\mathrm{e}^{-\mathrm{i}n\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Here, the two series in the integrand are rewritten by employing the 1D Poisson summation formula. Thus

$$\sum_{m\in\mathbf{Z}}(-1)^{mp}\,\mathrm{e}^{-\mathrm{i}m\xi}=2\pi\sum_{k\in\mathbf{Z}}\delta[\xi-(2k+p)\pi]$$

and

$$\sum_{n\in\mathbf{Z}}(-1)^{nq}\,\mathrm{e}^{-\mathrm{i}n\eta}=2\pi\sum_{l\in\mathbf{Z}}\delta[\eta-(2l+q)\pi].$$

Finally, on performing the required formal term-by-term integrations with regard to the properties of the δ-function we deduce

$$W_\nu(p,q;x)=\frac{4\pi^{2\nu-1}x^{-\nu}}{2^{2\nu}\Gamma(\nu)}\sum_{k\in\mathbf{Z}}\sum_{l\in\mathbf{Z}}^{(2k+p)^2+(2l+q)^2<4x^2/\pi^2}[4x^2/\pi^2-(2k+p)^2-(2l+q)^2]^{\nu-1}$$

where $\operatorname{Re}\nu>0$. Notice that this result and the 1D result given by equation (6.4) hold generally for real p and q provided that we define $(-1)^{mp}\equiv\exp(mp\pi\mathrm{i})$, $(-1)^{nq}\equiv\exp(nq\pi\mathrm{i})$.

The corresponding result for $R_\nu(p,q;x)$ obtained from equation (6.3) agrees with the special cases $R_\nu(x)$, $S_\nu(x)$, $T_\nu(x)$, $U_\nu(x)$ obtained earlier.

In concluding, we remark that different approaches to the summation of m-dimensional Schlömilch series may be found in [10] via L Schwartz's distributions and in [11] via induction.

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